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ON FUNCTIONAL EQUATIONS OF DEGENERATE DILOGARITHM

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ABSTRACT. Recently, the degenerate polylogarithm is introduced by Kim-Kim as a degenerate version of the polylogarithm. In this note, we derive some interesting functional equations related to degenerate dilogarithm.

1. INTRODUCTION

For $s \in \mathbb{C}$, we note that the polylogarithm is defined by a power series in z as

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s} + \cdots, \quad (\text{see } [1,3]). \tag{1}$$

This definition is valid for arbitrary complex order s and for all complex arguments z with $|z| < 1$; it can be extended to $|z| \geq 1$ by analytic continuation.

For any $\lambda \in \mathbb{R}$, the degenerate exponential functions are given by

$$e_\lambda^x(t) = \sum_{k=0}^{\infty} (x)_{k,\lambda} \frac{t^k}{k!}, \quad (\text{see } [2]), \tag{2}$$

where $(x)_{0,\lambda} = 1,\ (x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda), \quad (n \geq 1)$. Note that $\lim_{\lambda \to 0} e_\lambda^x(t) = e^{xt}$.

In particular, for $x = 1$, we use the notation

$$e_\lambda(t) = e_\lambda^1(t) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}}{n!} t^n.$$

The compositional inverse of $e_\lambda(t)$ is denoted by $\log_\lambda(t)$ and called the degenerate logarithm. Then we have

$$\log_\lambda(1+t) = \sum_{n=1}^{\infty} \lambda^{n-1} (1)_{n,\frac{1}{\lambda}} \frac{t^n}{n!} = \frac{1}{\lambda}\big((1+t)^\lambda - 1\big), \quad (\text{see } [2]). \tag{3}$$

Note that

$$\lim_{\lambda \to 0} \log_\lambda(1+t) = \log(1+t).$$

In [2], the degenerate polylogarithm is defined by

$$\mathrm{Li}_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1}(1)_{n,\frac{1}{\lambda}}}{(n-1)!n^k} x^n, \quad (|x| < 1,\ k \in \mathbb{Z}). \tag{4}$$

Note that

$$\lim_{\lambda \to 0} \mathrm{Li}_{k,\lambda}(x) = \mathrm{Li}_k(x), \quad (k \in \mathbb{Z}).$$

When $k = 2$, the degenerate polylogarithm is called the degenerate dilogarithm and given by

$$\mathrm{Li}_{2,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{(n-1)!n^2} x^n, \quad (|x| < 1). \tag{5}$$

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We note that

$$\lim_{\lambda\to 0}\mathrm{Li}_{2,\lambda}(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}=\mathrm{Li}_2(x),$$

where $\mathrm{Li}_2(x)$ is the dilogarithm (see [3]).

It is Carlitz who initiated a study of degenerate versions of Bernoulli and Euler polynomials and numbers. Studying degenerate versions of various special polynomials and numbers became an active area of research and yielded many interesting arithmetic and combinatorial results. In [2], the degenerate polylogarithm is introduced recently as a degenerate version of the polylogarithm. And the degenerate poly-Bernoulli polynomials and numbers, which is a new type of degenerate Bernoulli polynomials and numbers, are defined in terms of the degenerate polylogarithm and several properties concerning the degenerate poly-Bernoulli numbers are derived. Here we deduce several functional equations in the special case of the degenerate dilogarithm.

2. Functional equations of degenerate dilogarithm

From (5), we note that

$$(6)\qquad -\int_0^z\frac{\log_\lambda(1-t)}{t}dt=\sum_{n=1}^{\infty}\frac{(-\lambda)^{n-1}(1)_{n,\frac{1}{\lambda}}}{(n-1)!}\frac{1}{n}\int_0^z t^{n-1}dt$$
$$=\sum_{n=1}^{\infty}\frac{(-\lambda)^{n-1}(1)_{n,\frac{1}{\lambda}}}{(n-1)!n^2}z^n=\mathrm{Li}_{2,\lambda}(z).$$

Therefore, by (6), we obtain the following theorem.

Theorem 1. $\mathrm{Li}_{2,\lambda}(z)=-\int_0^z\frac{\log_\lambda(1-t)}{t}dt.$

Now, we observe that

$$(7)\qquad \mathrm{Li}_2(1-z)+\mathrm{Li}_{2,\lambda}(z)=-\int_0^{1-z}\frac{\log_\lambda(1-t)}{t}dt-\int_0^z\frac{\log_\lambda(1-t)}{t}dt$$
$$=-\int_0^{1-z}\frac{\log_\lambda(1-t)}{t}dt-\int_{1-z}^1\frac{\log_\lambda(1-t)}{t}dt-\int_0^z\frac{\log_\lambda(1-t)}{t}dt+\int_{1-z}^1\frac{\log_\lambda(1-t)}{t}dt$$
$$=-\int_0^1\frac{\log_\lambda(1-t)}{t}dt-\int_0^z\frac{\log_\lambda(1-t)}{t}dt+\int_0^z\frac{\log_\lambda t}{1-t}dt$$
$$=\mathrm{Li}_{2,\lambda}(1)-\int_0^z\left(\frac{\log_\lambda(1-t)}{t}-\frac{\log_\lambda t}{1-t}\right)dt.$$

We observe that

$$(8)\qquad \int_0^z\frac{\log_\lambda(1-t)}{t}dt=\Big[\log t\log_\lambda(1-t)\Big]_0^z+\int_0^z(1-t)^{\lambda-1}\log t\,dt$$
$$=\log z\log_\lambda(1-z)+\int_0^z(1-t)^{\lambda-1}\log t\,dt,$$

and

$$(9)\qquad \int_0^z\frac{\log_\lambda t}{1-t}dt=-\Big[\log_\lambda(t)\log(1-t)\Big]_0^z+\int_0^z t^{\lambda-1}\log(1-t)\,dt$$
$$=-\log_\lambda z\log(1-z)+\int_0^z t^{\lambda-1}\log(1-t)\,dt.$$

From (8) and (9), we have

$$\int_0^z \left(\frac{\log_\lambda(1-t)}{t} - \frac{\log_\lambda t}{1-t} \right) dt = \log z \log_\lambda(1-z) + \log(1-z)\log_\lambda z + \int_0^z (1-t)^{\lambda-1}\log t\, dt - \int_0^z t^{\lambda-1}\log(1-t)\, dt. \tag{10}$$

By (7) and (10), we get

$$\mathrm{Li}_{2,\lambda}(1-z) + \mathrm{Li}_{2,\lambda}(z) = \mathrm{Li}_{2,\lambda}(1) - \log z \log_\lambda(1-z) - \log_\lambda(z)\log(1-z) + \int_0^z \Big(-(1-t)^{\lambda-1}\log t + t^{\lambda-1}\log(1-t) \Big) dt. \tag{11}$$

Hence, by (11), we get the following theorem.

Theorem 2.

$$\int_0^z \big(t^{\lambda-1}\log(1-t) - (1-t)^{\lambda-1}\log t \big) dt = \mathrm{Li}_{2,\lambda}(1-z) + \mathrm{Li}_{2,\lambda}(z) - \mathrm{Li}_{2,\lambda}(1) + \log z \log_\lambda(1-z) + \log_\lambda(z)\log(1-z).$$

Now, we observe that

$$\begin{aligned} \int_0^z (1-t)^{\lambda-1}\log t\, dt &= \int_0^z (1-t)^{\lambda-1}\log(t-1+1)\, dt \\ &= -\sum_{n=1}^\infty \frac{1}{n}\int_0^z (1-t)^{n+\lambda-1} dt = -\sum_{n=1}^\infty \frac{1}{n}\left[-\frac{(1-t)^{n+\lambda}}{n+\lambda} \right]_0^z \\ &= \sum_{n=1}^\infty \frac{1}{n(n+\lambda)}\big((1-z)^{n+\lambda} - 1 \big). \end{aligned} \tag{12}$$

and

$$\int_0^z t^{\lambda-1}\log(1-t)\, dt = -\sum_{n=1}^\infty \frac{1}{n}\int_0^z t^{n+\lambda-1} dt = -\sum_{n=1}^\infty \frac{z^{n+\lambda}}{n(n+\lambda)}. \tag{13}$$

From (12) and (13), we note that

$$\begin{aligned} &\int_0^z \Big(t^{\lambda-1}\log(1-t) - (1-t)^{\lambda-1}\log t \Big) dt \\ &= -\sum_{n=1}^\infty \frac{z^{n+\lambda}}{n(n+\lambda)} - \sum_{n=1}^\infty \frac{1}{n(n+\lambda)}\Big((1-z)^{n+\lambda} - 1 \Big) \\ &= \sum_{n=1}^\infty \frac{1}{n(n+\lambda)}\Big(1 - (1-z)^{n+\lambda} - z^{n+\lambda} \Big). \end{aligned} \tag{14}$$

Therefore, by Theorem 2 and (14), we obtain the following theorem,

Theorem 3. *For $\lambda \in \mathbb{R}$, we have*

$$\sum_{n=1}^\infty \frac{1}{n(n+\lambda)}\Big(1 - (1-z)^{n+\lambda} - z^{n+\lambda} \Big) = \mathrm{Li}_{2,\lambda}(1-z) + \mathrm{Li}_{2,\lambda}(z) - \mathrm{Li}_{2,\lambda}(1) + \log z \log_\lambda(1-z) + \log_\lambda(z)\log(1-z).$$

From Theorem 1, we have

$$
\begin{aligned}
(15)\qquad & \mathrm{Li}_{2,\lambda}\left(-\frac{z}{1-z}\right)=\mathrm{Li}_{2,\lambda}\left(1-\frac{1}{1-z}\right)\\
&=-\int_{0}^{1-\frac{1}{1-z}}\frac{1}{t}\log_{\lambda}(1-t)\,dt=-\int_{1}^{1-z}\frac{u}{u-1}\left(\log_{\lambda}\frac{1}{u}\right)\frac{1}{u^{2}}du\\
&=\int_{1}^{1-z}\frac{1}{(u-1)u}\log_{-\lambda}(u)\,du=\int_{1}^{1-z}\left(\frac{1}{u-1}-\frac{1}{u}\right)\log_{-\lambda}(u)\,du\\
&=-\int_{1}^{1-z}\frac{1}{1-u}\log_{-\lambda}(u)\,du-\int_{1}^{1-z}\frac{1}{u}\log_{-\lambda}(u)\,du\\
&=\int_{0}^{z}\frac{\log_{-\lambda}(1-t)}{t}dt-\int_{1}^{1-z}\frac{\log_{-\lambda}(t)}{t}dt\\
&=-\mathrm{Li}_{2,-\lambda}(z)-\int_{1}^{1-z}\frac{1}{t}\log_{-\lambda}(t)dt.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(16)\qquad & \int_{1}^{1-z}\frac{1}{t}\log_{-\lambda}(t)dt=\Big[\log t\log_{-\lambda}(t)\Big]_{1}^{1-z}-\int_{1}^{1-z}\frac{\log t}{t}t^{-\lambda}dt\\
&=\log(1-z)\log_{-\lambda}(1-z)-\int_{1}^{1-z}\frac{\log t}{t}(t^{-\lambda}-1+1)dt\\
&=\log(1-z)\log_{-\lambda}(1-z)+\lambda\int_{1}^{1-z}\frac{\log t}{t}\log_{-\lambda}(t)dt-\int_{1}^{1-z}\frac{\log t}{t}dt.\\
&=\log(1-z)\log_{-\lambda}(1-z)+\lambda\int_{1}^{1-z}\frac{\log t}{t}\log_{-\lambda}(t)dt-\frac{1}{2}\big(\log(1-z)\big)^{2}.
\end{aligned}
$$

By (15) and (16), we get the following result.

Theorem 4. *For $\lambda\in\mathbb{R}$, we have*

$$
\begin{aligned}
&-\lambda\int_{1}^{1-z}\frac{\log t}{t}\log_{-\lambda}(t)dt\\
&=\mathrm{Li}_{2,\lambda}\left(-\frac{z}{1-z}\right)+\mathrm{Li}_{2,\lambda}(z)+\log(1-z)\cdot\log_{-\lambda}(1-z)-\frac{1}{2}\big(\log(1-z)\big)^{2}.
\end{aligned}
$$

By replacing z by $1-x$, we get

$$
\begin{aligned}
&-\lambda\int_{1}^{x}\frac{\log t}{t}\log_{-\lambda}(t)dt\\
&=\mathrm{Li}_{2,\lambda}\left(1-\frac{1}{x}\right)+\mathrm{Li}_{2,\lambda}(1-x)+\log x\log_{-\lambda}(x)-\frac{1}{2}\big(\log x\big)^{2}.
\end{aligned}
$$

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